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The supersymmetric $t-J$ model in one dimension

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Abstract. The $t-J$ model (related to the strong-correlation limit of the Hubbard model) is shown to be soluble in one dimension using the Bethe ansatz. The solution holds only when the Hamiltonian is supersymmetric. The ground state in the presence of holes is found to be gapless, and to have no magnetization.

1. Introduction

There is a strong belief (Anderson 1988, Fukuyama *et al* 1989) that electron correlations are important in distinguishing the new high-temperature superconductors from conventional ones. The existence of antiferromagnetism in the absence of doping for the new materials is evidence for this. Hubbard (1963) was very influential in the study of such correlations. He proposed a lattice Hamiltonian,

$$H=\sum_{\substack{\langle ij\rangle\\ \sigma}} tc^{\dagger}_{i\sigma}c_{j\sigma}+\frac{U}{2}\sum_{i,\sigma} n_{i,\sigma}\sigma_{i,-\sigma} \tag{1}$$

as an example of a system which clearly accommodates the atomic ($t/U\to 0$) and band theory ($U/t\to 0$) limits. i and j are nearest-neighbour sites and $c_{j\sigma}$ destroys an electron with z-component of spin σ at site j. $n_{i,\sigma}$ is $c^{\dagger}_{i\sigma}c_{i\sigma}$, the number operator. Only the low-energy states can have any possible relevance to superconductivity. Since we are interested in the strong correlation ($U/t\gg 1$) limit we may take

$$H_0=\frac{U}{2}\sum_{i,\sigma} n_{i,\sigma}n_{i,-\sigma} \tag{2}$$

for the unperturbed Hamiltonian. For a lattice with N sites and $(N-n)$ electrons the ground state of H_0 cannot have more than one electron per site. There is a set of

$$\binom{N}{N-n}2^{N-n}$$

degenerate ground states of H_0. From degenerate perturbation theory (Lindgren and Morrison 1986, Pike *et al* 1991) we can construct an effective Hamiltonian $H_{\rm eff}$ which operates on this set but has the same low energy spectrum as H. Since the Hilbert

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space of H_{eff} is much smaller than that for H this 'effective' description is very economical. The resultant H_{eff} is

$$H_{\mathrm{eff}} = P\Bigg(t \sum_{\substack{\langle ij\rangle \\ \sigma}} c^{\dagger}_{i\sigma} c_{j\sigma} + J \sum_{\langle ij\rangle} (\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \tfrac{1}{4}) n_i n_j - \frac{J}{2} \sum_{\substack{\langle ikj\rangle \\ \sigma}} (c^{\dagger}_{i\sigma} n_{k,-\sigma} c_{j\sigma} + c^{\dagger}_{i,-\sigma} c_{k,-\sigma} c^{\dagger}_{k\sigma} c_{j\sigma}) \Bigg) P \tag{3}$$

where P is the projection operator onto the set of ground states of H_0, $J = 2t^2/U(\ll t)$, $\boldsymbol{S}_i$ is a spin operator and site k is a nearest neighbour to sites i and j. The connection of H and H_{eff} with the copper oxide-based high-temperature superconductors is not obvious. Indeed it is generally accepted that there is substantial overlap of the electron orbitals on copper and oxygen. This would naturally lead to an H also involving creation and annihilation operators for oxygen orbitals. Zhang and Rice (1988) nonetheless showed that an effective model similar to (3) can emerge where $J \leqslant t$. Further support for the validity of this model with parameter values as large as $J \sim 1.53t$ has been given recently (Jefferson 1990).

For $n/N \ll 1$ the three-site terms in H_{eff} are small compared with the other terms and are often ignored. The resulting H_{eff} is called for obvious reasons the $t-J$ Hamiltonian (H_{t-J}). In a recent letter (Sarkar 1990a) the method of solution for this model in one dimension where $J/2t = 1$ has been indicated. At this point in parameter space H_{eff} is invariant under the group of transformations of a supergroup U(1/2) (Wiegmann 1988, Cornwell 1989, Sarkar 1990b). Details of this solution which uses the Bethe ansatz (Bethe 1931) will now be given. In particular we will find the ground state and excited state energies as a function of concentration near half filling. This solution is not a simple consequence of the Bethe ansatz solution of the Hubbard model (Lieb and Wu 1968) for two reasons. Firstly the large U/t limit of the Hubbard model has three-site terms and secondly J is not very much less than t.

2. Supersymmetry

H_{t-J} operates on a Hilbert space which is spanned by states of the form

$$\bigotimes_i |\alpha_i\rangle$$

where $\otimes$ denotes a direct product and $|\alpha_i\rangle$ is $|0\rangle$, $|\downarrow\rangle$ or $|\uparrow\rangle$. $|\downarrow\rangle$ and $|\uparrow\rangle$ represent Wannier states with spin down and spin up respectively. $|0\rangle$ is a hole state. With the basis $|\alpha_i\rangle$ it is natural to associate operators

$$X_i^{\alpha\beta} = |\alpha_i\rangle\langle\beta_i|. \tag{4}$$

Now $X_i^{0\uparrow}$ has a fermionic nature since it destroys an up-spin electron whereas $X_i^{\uparrow\downarrow}$ is bosonic. This leads naturally to an operator algebra involving both commutators and anticommutators, the latter occurring only if both operators are fermionic. The resulting so-called superalgebra is

$$[X_i^{\alpha\beta}, X_j^{\alpha'\beta'}]_{\pm} = \delta_{ij}(X_i^{\alpha\beta'}\delta_{\beta\alpha'} \pm X_i^{\alpha'\beta}\delta_{\beta'\alpha}) \tag{5}$$

where $[\,,]_+$ is an anticommutator and $[\,,]_-$ is a commutator. It is standard (Bars and Günaydin 1983) to represent these operators in terms of a bosonic and two fermionic harmonic oscillators, e.g.

$$X_i^{0\sigma} = f_i^\sigma b_i^\dagger \qquad X_i^{\sigma\sigma'} = f_i^{\sigma\dagger} f_i^{\sigma'} \qquad X_i^{00} = b_i^\dagger b_i \tag{6}$$

where

$$[f_i^\sigma, f_j^{\dagger\sigma'}]_+ = \delta_{\sigma\sigma'}\delta_{ij} \qquad [f_i^\sigma, f_j^{\sigma'}]_+ = [f_i^{\sigma\dagger}, f_j^{\sigma'\dagger}]_+ = 0 \tag{7}$$

and

$$[b_i, b_j^\dagger]_- = \delta_{ij} \qquad [b_i, b_j]_- = [b_i^\dagger, b_j^\dagger]_- = 0. \tag{8}$$

This representation will be useful later.

Using the X-operators it is possible to rewrite H_{t-J} without the formal use of projection operators P. Since we are dealing with spin-$\frac{1}{2}$ we can write

$$J(\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \tfrac{1}{4}) = \frac{J}{4}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1) \tag{9}$$

where $\boldsymbol{\sigma}_i$ are the Pauli spin matrices at site i. This can be further rewritten as

$$\frac{J}{2}(P_{ij} - 1)$$

where

$$P_{ij} = \tfrac{1}{2}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + 1). \tag{10}$$

It is easy to verify that

$$P_{ij}|\sigma\rangle_i|\sigma'\rangle_j = |\sigma'\rangle_i|\sigma\rangle_j \tag{11}$$

so that P_{ij} is a permutation on spin labels. In terms of the X-operators:

$$P_{i\,i+1} = \sum_{\sigma,\sigma'} X_i^{\sigma\sigma'} X_{i+1}^{\sigma'\sigma}. \tag{12}$$

We need to write $\delta_{n_i 1}\delta_{n_{i+1} 1}$ (where $n_i = n_{i\uparrow} + n_{i\downarrow}$) in terms of X-operators as well. We first note that

$$\sum_i \delta_{n_i 0} = \sum_i \delta_{n_i 0}\delta_{n_{i+1} 0} + \tfrac{1}{2}\sum_i (\delta_{n_i 0}\delta_{n_{i+1} 1} + \delta_{n_i 1}\delta_{n_{i+1} 0}) \tag{13}$$

$$\sum_i (\delta_{n_i 0} + \delta_{n_i 1}) = N \tag{14}$$

and

$$\sum_i (\delta_{n_i 0}\delta_{n_{i+1} 0} + \delta_{n_i 1}\delta_{n_{i+1} 1} + \delta_{n_i 0}\delta_{n_{i+1} 1} + \delta_{n_i 1}\delta_{n_{i+1} 0}) = N \tag{15}$$

where N is the number of lattice sites. The validity of (13) and (15) is best established by examining examples. From (13) and (15) we have

$$2\sum_i (\delta_{n_i 0} - \delta_{n_i 0}\delta_{n_{i+1} 0}) + \sum_i \delta_{n_i 1}\delta_{n_{i+1} 1} + \sum_i \delta_{n_i 0}\delta_{n_{i+1} 0} = N \tag{16}$$

and on using (14) we deduce

$$\sum_i \delta_{n_i 1}\delta_{n_{i+1} 1} = \sum_i \delta_{n_i 0}\delta_{n_{i+1} 0} - \sum_i \delta_{n_i 0} + \sum_i \delta_{n_i 1}. \tag{17}$$

Since

$$\sum_i \delta_{n_i 0}\delta_{n_{i+1} 0}=\sum_i X_i^{00}X_{i+1}^{00} \tag{18}$$

(and $\Sigma_i\, \delta_{n_i 0}$ and $\Sigma_i\, \delta_{n_i 1}$ are constants) we can write

$$\frac{J}{2}\sum_i (P_{i\,i+1}-1)=\frac{J}{2}\sum_i X_i^{\sigma\sigma'}X_{i+1}^{\sigma'\sigma}-\frac{J}{2}\sum_i X_i^{00}X_{i+1}^{00} \tag{19}$$

up to an additive constant. (A summation convention is understood.) A Hamiltonian H' which is more general than H_{t-J} may then be written as

$$H'=g\sum_i (X_i^{\sigma 0}X_{i+1}^{0\sigma}+X_{i+1}^{\sigma 0}X_i^{0\sigma})+g'\sum_i X_i^{\sigma\sigma'}X_{i+1}^{\sigma'\sigma}+g''\sum_i X_i^{00}X_{i+1}^{00} \tag{20}$$

g, g' and g'' being constants.

H_{t-J} is obtained when

$$g=-t \qquad g'=\frac{J}{2}=-g'' \tag{21}$$

(provided we make the canonical transformation $b_i \to (-1)^i b_i$ in (6)).

The generators $X^{0\sigma}$ $(=\Sigma\, X_i^{0\sigma})$, $X^{\sigma 0}$ $(=\Sigma\, X_i^{\sigma 0})$ and $X^{\sigma\sigma'}$ $(=\Sigma\, X_i^{\sigma\sigma'})$ form a superalgebra isomorphic to the single site superalgebra. The bosonic generators $X^{\sigma\sigma'}$ commute with the Hamiltonian for arbitrary g, g' and g''. For H' to be supersymmetric (with respect to this superalgebra), it remains to check that the condition

$$\left[\sum_j X_j^{0\sigma}, H'\right]_-=0 \tag{22}$$

is satisfied. Now

$$\left[\sum_j X_j^{0\sigma}, g\sum_i (X_i^{\sigma'0}X_{i+1}^{0\sigma'}+X_{i+1}^{\sigma'0}X_i^{0\sigma'})\right]_-$$
$$=g\sum_i (X_i^{00}X_{i+1}^{0\sigma}+X_i^{\sigma'\sigma}X_{i+1}^{0\sigma'}+X_{i+1}^{00}X_i^{0\sigma}+X_{i+1}^{\sigma'\sigma}X_i^{0\sigma'}) \tag{23}$$

$$\left[\sum_j X_j^{0\sigma}, g'\sum_i X_i^{\sigma'\sigma''}X_{i+1}^{\sigma''\sigma'}\right]_-=g'\sum_i (X_i^{0\sigma'}X_{i+1}^{\sigma'\sigma}+X_i^{\sigma'\sigma}X_{i+1}^{0\sigma'}) \tag{24}$$

and

$$\left[\sum_j X_j^{0\sigma}, g''\sum_i X_i^{00}X_{i+1}^{00}\right]_-=-g''\sum_i (X_i^{0\sigma}X_{i+1}^{00}+X_i^{00}X_{i+1}^{0\sigma}) \tag{25}$$

and so

$$\left[\sum_j X_j^{0\sigma}, H'\right]_-=(g-g'')\sum_i (X_i^{00}X_{i+1}^{0\sigma}+X_{i+1}^{00}X_i^{0\sigma})+(g+g')\sum_i (X_i^{\sigma'\sigma}X_{i+1}^{0\sigma'}+X_i^{0\sigma'}X_{i+1}^{\sigma'\sigma}). \tag{26}$$

For (22) to hold we require

$$g=g''=-g' \tag{27}$$

which we will call the supersymmetric $t-J$ model. In the next section we will show how the resulting Hamiltonian can be interpreted as a generalized permutation operator.

Owing to the predominance of supersymmetry applications in a particle physics context it is often assumed that the supersymmetric algebra contains the Poincaré algebra as a subalgebra (Cornwell 1989, p 79). Consequently the Hamiltonians can be expressed in terms of bilinears in the fermionic generators (e.g. see de Crombrugghe and Rittenberg 1983). The $t-J$ Hamiltonian is not of this kind and the supersymmetry is of a kinematic type. Our use of the term 'supersymmetry' is similar to that adopted quite commonly in, for example, nuclear physics (Iachello 1985).

3. Generalized permutation operator

In terms of the harmonic oscillator representation the hopping term in H_{t-J} is

$$t\sum_i (b_i b_{i+1}^\dagger f_i^{\sigma\dagger} f_{i+1}^\sigma + b_{i+1} b_i^\dagger f_{i+1}^{\sigma\dagger} f_i^\sigma).$$

As a consequence each site of the lattice is occupied by the boson or a fermion (either an up or down fermion). The constraint on the Hilbert space of no double occupancy is

$$b_i^\dagger b_i + f_{i\sigma}^\dagger f_{i\sigma} = 0. \tag{28}$$

A generalized permutation operator will interchange fermions and bosons (as well as interchange just fermions) with the same amplitude. The relevance of such operators will now be discussed. The permutation aspect of the Heisenberg term has already been discussed and was in fact noticed by Bethe (1931). The hopping term is also a permutation operator but now between bosons and fermions. We will examine this aspect through an example. A lattice with $(N-4)$ up spins, two down spins and two hole has a state $|\psi\rangle$ of the form

$$|\psi\rangle = \sum_{i,j,k,l} a(i,j,k,l)\dots b_i^\dagger \dots b_j^\dagger \dots f_k^{\downarrow\dagger} \dots f_l^{\downarrow\dagger} \dots |0\rangle \tag{29}$$

where ... denote creation operators for up spins. For definiteness we consider the part of $|\psi\rangle$

$$a(i,i+1,k,k+1)\dots b_i^\dagger b_{i+1}^\dagger f_{i+2}^{\uparrow\dagger} \dots f_{k-1}^{\uparrow\dagger} f_k^{\downarrow\dagger} f_{k+1}^{\downarrow\dagger} f_{k+2}^{\uparrow\dagger} \dots |0\rangle$$

and operate on it with

$$t(b_{i+1} b_{i+2}^\dagger f_{i+1}^{\sigma\dagger} f_{i+2}^\sigma + b_{i+2} b_{i+1}^\dagger f_{i+2}^{\sigma\dagger} f_{i+1}^\sigma)$$

a part of H_{t-J}. Now

$$\dots b_{i+1} b_{i+2}^\dagger f_{i+1}^{\uparrow\dagger} f_{i+2}^\uparrow b_i^\dagger b_{i+1}^\dagger f_{i+2}^{\uparrow\dagger} \dots |0\rangle = \dots b_{i+1} b_{i+2}^\dagger b_i^\dagger b_{i+1}^\dagger f_{i+1}^{\uparrow\dagger} f_{i+2}^\uparrow f_{i+2}^{\uparrow\dagger} \dots |0\rangle \tag{30}$$

and

$$f_{i+1}^{\uparrow\dagger} f_{i+2}^\uparrow f_{i+2}^{\uparrow\dagger} = f_{i+1}^{\uparrow\dagger}(1 - f_{i+2}^{\uparrow\dagger} f_{i+2}^\uparrow). \tag{31}$$

The last term in (31) when pulled through in (30) gives zero. Consequently (30) becomes

$$\dots b_{i+1} b_{i+2}^\dagger b_i^\dagger b_{i+1}^\dagger f_{i+1}^{\uparrow\dagger} \dots |0\rangle = \dots b_{i+2}^\dagger b_i^\dagger f_{i+1}^{\uparrow\dagger} \dots |0\rangle = \dots b_i^\dagger f_{i+1}^{\uparrow\dagger} b_{i+2}^\dagger \dots |0\rangle. \tag{32}$$

The hole at $(i+1)$ and the up spin at $(i+2)$ have thus been swapped. This is just the effect of a permutation operator which will be denoted by $P_{i+1,i+2}^{(0,\uparrow)}$. The same result is found by examining other cases. H_{t-J} can then be written as

$$H_{t-J} = t\sum_{i,\sigma} P_{i\,i+1}^{(0,\sigma)} + \frac{J}{2}\sum_i (P_{i\,i+1} - P_{i\,i+1}^{(0,0)}) \tag{33}$$

with

$$P_{i\,i+1}^{(0,0)}(\ldots b_i^{\dagger} b_{i+1}^{\dagger} \ldots)|0\rangle = (\ldots b_{i+1}^{\dagger} b_i^{\dagger} \ldots)|0\rangle. \tag{34}$$

The arguments above easily generalize to a situation when there are more 'flavours' of fermions.

The permutation symmetry of the ground state can be determined without a detailed calculation of energies for different Young tableau representations. The argument has been essentially given by Lai and Yang (1971) who restricted themselves to two flavours but their reasoning applies also to the case of more flavours. If there is an odd number N^i of fermions of flavour i ($i = 1, \ldots, m$) and N^0 bosons (with $\Sigma_i N^i + N^0 = N$) then the permutation symmetry of the ground state is given by the Young tableau

$$(m + N^0, m^{N_m - 1}, (m-1)^{N_{m-1} - N_m}, (m-2)^{N_{m-2} - N_{m-1}}, \ldots, 1^{(N_1 - N_2)}). \tag{35}$$

The precise nature of the Hamiltonian played no role in the discussion of Lai and Yang. They dealt with a continuum and allowed double occupation at a site. The continuum aspect was not relevant to their argument while the amplitude for double occupation can be made arbitrarily small by adding an energy penalty term to their Hamiltonian. Consequently the result of (35) is implied by Lai and Yang also for our case. It will be convenient to work with the conjugate Young tableau representation (Andrei *et al* 1983) which is equivalent to a canonical transformation on the variables (Sarkar 1990a).

4. Bethe ansatz

As an example of a simple case away from half-filling let us consider a lattice with $(N-2)$ up spins, one down spin and one hole. Any state $|\psi\rangle$ of this lattice has the form

$$|\psi\rangle = \sum_{x_1, x_2} \alpha(x_1, x_2) f_1^{\uparrow\dagger} f_2^{\uparrow\dagger} \ldots f_{x_1-1}^{\uparrow\dagger} f_{x_1}^{\downarrow\dagger} f_{x_1+1}^{\uparrow\dagger} \ldots b_{x_2}^{\dagger} \ldots f_N^{\uparrow\dagger}|0\rangle. \tag{36}$$

The down spin and hole are located at x_1 and x_2 respectively. We now demand that $|\psi\rangle$ is an energy eigenstate and so

$$H|\psi\rangle = E|\psi\rangle. \tag{37}$$

For x_1 and x_2 far apart

$$E\alpha(x_1, x_2) = \frac{-J}{2}(\alpha(x_1+1, x_2) + \alpha(x_1-1, x_2)) + t(\alpha(x_1, x_2-1) + \alpha(x_1, x_2+1)). \tag{38}$$

The Bethe ansatz is

$$\begin{aligned} \alpha(x_1, x_2) &= \alpha_Q(x_1, x_2) \\ &= A_1(Q)\exp(\mathrm{i}(k_1 x_{Q1} + k_2 x_{Q2})) + A_2(Q)\exp(\mathrm{i}(k_2 x_{Q1} + k_1 x_{Q2})) \end{aligned} \tag{39}$$

where Q is an element of S_2 the permutation group on two objects and defines a sector $x_{Q1} < x_{Q2}$. For brevity

$$Q = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

will be denoted by 1 and

$$Q=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$$

by 2. $A_1(Q)$, $A_2(Q)$, k_1, and k_2 are constants which need to be determined.

Equation (37) implies that

$$\begin{aligned}
&E(A_1(1)\exp[\mathrm{i}(k_1x_1+k_2x_2)]+A_2(1)\exp[\mathrm{i}(k_2x_1+k_1x_2)])\\
&\quad=-\frac{J}{2}(A_1(1)\exp\{\mathrm{i}[k_1(x_1+1)+k_2x_2]\}+A_2(1)\exp\{\mathrm{i}[k_2(x_1+1)+k_1x_2]\}\\
&\qquad+A_1(1)\exp\{\mathrm{i}[k_1(x_1-1)+k_2x_2]\}+A_2(1)\exp\{\mathrm{i}[k_2(x_1-1)+k_1x_2]\})\\
&\qquad+t(A_1(1)\exp\{\mathrm{i}[k_1x_1+k_2(x_2-1)]\}+A_2(1)\exp\{\mathrm{i}[k_2x_1+k_1(x_2-1)]\}\\
&\qquad+A_1(1)\exp\{\mathrm{i}[k_1x_1+k_2(x_2+1)]\}+A_2(1)\exp\{\mathrm{i}[k_2x_1+k_1(x_2+1)]\})\\
&\quad=2\left(-\frac{J}{2}\cos k_1+t\cos k_2\right)A_1(1)\exp[\mathrm{i}(k_1x_1+k_2x_2)]\\
&\qquad+2\left(-\frac{J}{2}\cos k_2+t\cos k_1\right)A_2(1)\exp[\mathrm{i}(k_2x_1+k_1x_2)].
\end{aligned}\tag{40}$$

(In (40) we have ignored an overall constant energy shift.) Hence a necessary condition for an energy eigenstate is

$$-\frac{J}{2t}=1. \tag{41}$$

This is also the supersymmetric condition (27). Henceforth we will choose units and phases so that $t=-1$ and also require (41) to hold. For the term in (36) proportional to

$$f_1^{\uparrow\dagger}f_2^{\uparrow\dagger}b_{x_1}^{\dagger}f_{x_1+1}^{\downarrow\dagger}f_{x_1+2}^{\uparrow\dagger}\dots f_N^{\uparrow\dagger}|0\rangle$$

the energy eigenstate condition gives

$$\begin{aligned}
&(2(\cos k_1+\cos k_2)+1)\alpha(x_1,x_1+1)\\
&\quad=\alpha(x_1+1,x_1)+\alpha(x_1,x_1+2)+\alpha(x_1-1,x_1+1)
\end{aligned}\tag{42}$$

and

$$\begin{aligned}
&(2(\cos k_1+\cos k_2)+1)\alpha(x_1+1,x_1)\\
&\quad=\alpha(x_1,x_1+1)+\alpha(x_1+1,x_1-1)+\alpha(x_1+2,x_1).
\end{aligned}\tag{43}$$

These two equations lead to

$$A_1(2)=u^{12}A_2(2)+v^{12}A_2(1) \qquad A_1(1)=u^{12}A_2(1)+v^{12}A_2(2) \tag{44}$$

where

$$u^{12}=-\frac{(1+\mathrm{e}^{\mathrm{i}k_1})(1+\mathrm{e}^{\mathrm{i}k_2})}{\mathrm{e}^{\mathrm{i}(k_1+k_2)}+2\,\mathrm{e}^{\mathrm{i}k_2}+1} \tag{45}$$

and

$$v^{12}=\frac{\mathrm{e}^{\mathrm{i}k_1}-\mathrm{e}^{\mathrm{i}k_2}}{\mathrm{e}^{\mathrm{i}(k_1+k_2)}+2\,\mathrm{e}^{\mathrm{i}k_2}+1}. \tag{46}$$

We need, of course, to be also able to consider situations other than that of (29). If there are $N^{\downarrow}$ down spins at $x_1, \ldots, x_{N^{\downarrow}}$ and N^0 holes at $x_{N^{\downarrow}+1}, \ldots, x_{N^{\downarrow}+N^0}$ then the Bethe ansatz is

$$\alpha(x_1, \ldots, x_{N^{\downarrow}+N^0}) = \sum_{\substack{P,Q \\ \in S_{N^{\downarrow}+N^0}}}^{\prime} A_P(Q) \exp\left(\mathrm{i} \sum_{j=1}^{N^{\downarrow}+N^0} k_{Pj} x_{Qj}\right) \theta(x_Q) \tag{47}$$

where $\theta(x_Q)$ denotes the region $x_{Q1} < x_{Q2} < \ldots < X_{Q(N^{\downarrow}+N^0)}$.

For the Bethe ansatz solution to work, entities such as u^{12} and v^{12} have to satisfy some identities. However we will not explicitly check these consistency conditions since we have shown elsewhere (Sarkar 1990a) that the supersymmetric $t-J$ model can be mapped onto a model of Lai (1974). The latter model has been shown by Lai to be soluble by the Bethe ansatz.

In order to proceed further, periodic boundary conditions have to be imposed on the wavefunction. This is by now a standard although somewhat complicated procedure (Yang 1967, Sutherland 1975, Lai and Yang 1971) and is known as the generalized Bethe hypothesis. It is discussed in detail by Andrei *et al* (1983) and so we will just give the results of the procedure. Apart from the ks involved in the Bethe ansatz, some auxiliary variables Λ appear which are related to a proper description of the permutation symmetry of $A_P(Q)$. The equations that emerge are

$$\mathrm{e}^{\mathrm{i}k_j N} = \prod_{\gamma=1}^{N^0} \left(\frac{\mathrm{i}(\Lambda_\gamma - \alpha_j) + \frac{1}{2}}{\mathrm{i}(\Lambda_\gamma - \alpha_j) - \frac{1}{2}}\right) \prod_{k=1}^{N^{\downarrow}+N^0} \frac{\mathrm{i}(\alpha_j - \alpha_k) + 1}{\mathrm{i}(\alpha_j - \alpha_k) - 1} \tag{48}$$

and

$$\prod_{j=1}^{N^{\downarrow}+N^0} \left(\frac{\mathrm{i}(\Lambda_\delta - \alpha_j) + \frac{1}{2}}{\mathrm{i}(\Lambda_\delta - \alpha_j) - \frac{1}{2}}\right) = 1 \tag{49}$$

where $\alpha_j = \frac{1}{2} \tan \frac{1}{2} k_j$. It is customary to take the logarithm of these equations. If

$$\theta(x) = -2 \tan^{-1} x \tag{50}$$

then

$$\mathrm{e}^{\mathrm{i}\theta} = \frac{1 - \mathrm{i}x}{1 + \mathrm{i}x}. \tag{51}$$

On taking logarithms of both sides of (51) we get

$$\theta = -\mathrm{i} \log\left(\frac{1 - \mathrm{i}x}{1 + \mathrm{i}x}\right) + 2\pi J \tag{52}$$

where J is an integer. Using these elementary facts, (48) and (49) give

$$N k_j = 2\pi J_j - \sum_{k=1}^{N^{\downarrow}+N^0} \theta(\alpha_j - \alpha_k) + \sum_{\gamma=1}^{N^0} \theta(2\alpha_j - 2\Lambda_\gamma) \tag{53}$$

$$\sum_{j=1}^{N^{\downarrow}+N^0} \theta(2\Lambda_\gamma^{1} - 2\alpha_j) + 2\pi J_\gamma^{1} = 0. \tag{54}$$

These equations readily generalize to the m fermion flavour case alluded to earlier. In the generalized Bethe hypothesis if there are m fermion flavours then there are $(m-1)$

flavours of Λ. The resulting equations are

$$Nk_j = 2\pi J_j - \sum_{k=1}^{N-N^1} \theta(\alpha_j - \alpha_k) + \sum_{\gamma=1}^{N-N^1-N^2} \theta(2\alpha_j - 2\Lambda^1_\gamma)$$

$$\sum_{\gamma'=1}^{N-\Sigma^r_{i=1}N^i} \theta(\Lambda^{r-1}_\gamma - \Lambda^{r-1}_{\gamma'}) \qquad ,$$

$$= 2\pi J^{r-1}_\gamma + \sum_{\gamma'=1}^{N-\Sigma^{r-1}_{i=1}N^i} \theta(2\Lambda^{r-1}_\gamma - 2\Lambda^{r-2}_{\gamma'})$$

$$+ \sum_{\varepsilon=1}^{N-\Sigma^{r+1}_{j=1}N^j} \theta(2\Lambda^{r-1}_\gamma - 2\Lambda^r_\varepsilon) \qquad (2 \leqslant r \leqslant m-1) \tag{55}$$

and

$$\sum_{\gamma=1}^{N-\Sigma^m_{j=1}N^j} \theta(2\Lambda^{m-1}_\varepsilon - 2\Lambda^{m-2}_\gamma) + 2\pi J^{m-1}_\varepsilon = 0.$$

The r on Λ^r is the flavour index and the range of γ is $1 \leqslant \gamma \leqslant (N - \Sigma^{r+1}_{j=1} N^j)$.

We will leave further discussion of the m flavour case for elsewhere. For any lattice of macroscopic size, (53) and (54) are too complicated. Consequently we will follow the customary practice and convert to a set of coupled integral equations. The reasonable assumption is made that

$$\alpha_{j+1} - \alpha_j \sim \mathrm{O}\left(\frac{1}{N}\right) \tag{56}$$

and

$$\Lambda_{\gamma+1} - \Lambda_\gamma \sim \mathrm{O}\left(\frac{1}{N}\right).$$

From (53)

$$2\pi(J_{j+1} - J_j) = \sum_{k=1}^{N-N^1} (\theta(\alpha_{j+1} - \alpha_k) - \theta(\alpha_j - \alpha_k))$$

$$- \sum_{\gamma=1}^{N-N^1-N^2} (\theta(2\alpha_{j+1} - 2\Lambda^1_\gamma) - \theta(2\alpha_j - 2\Lambda^1_\gamma))$$

$$+ 2N[\tan^{-1}(2\alpha_{j+1}) - \tan^{-1}(2\alpha_j)]. \tag{57}$$

The general experience with the Bethe ansatz shows that for the ground state $J_{j+1} - J_j = 1$. Excited states appear when there are j' such that $J_{j'+1} - J_{j'} = 2$. J' is called a hole. Hence

$$2\pi\left(1 - \sum_{j'} \delta_{jj'}\right) + 2\pi 2 \sum_{j'} \delta_{jj'}$$

$$\simeq \sum_{k=1}^{N-N^1} \theta'(\alpha_j - \alpha_k)(\alpha_{j+1} - \alpha_j) \sum_{\gamma=1}^{N-N^1-N^2} \theta'(2\alpha_j - 2\Lambda^1_\gamma) 2(\alpha_{j+1} - \alpha_j)$$

$$+ 2N \frac{1}{1+4\alpha_j^2} 2(\alpha_{j+1} - \alpha_j). \tag{58}$$

If N_h is the number of holes then

$$\sum_{j=1}^{N_\mathrm{h}} \delta(\alpha - \alpha^\mathrm{h}_j) + \rho_1(\alpha) = \frac{1}{N} \sum_{k=1}^{N-N_1} \theta'(\alpha - \alpha_k) - \frac{2}{N} \sum_{\gamma=1}^{N-N^1-N^2} \theta'(2\alpha - 2\Lambda^1_\gamma) + \frac{4}{1+4\alpha^2} \tag{59}$$

where

$$\rho_1(\alpha_j) = \frac{2\pi}{N} \frac{1}{(\alpha_{j+1} - \alpha_j)}. \tag{60}$$

In the $N\to\infty$ limit ρ_1 is a distribution and the sums on the right-hand side of (59) become integrals, and consequently

$$\rho_1(\alpha)+\frac{2\pi}{N}\sum_{j=1}^{N_h}\delta(\alpha-\alpha_j^h)$$
$$=-\frac{1}{\pi}\int\frac{1}{1+(\alpha-\alpha')^2}\rho_1(\alpha')\,\mathrm{d}\alpha'$$
$$+\frac{2}{\pi}\int\frac{1}{1+4(\alpha-\Lambda^1)^2}\rho_2(\Lambda^1)\,\mathrm{d}\Lambda^1+\frac{4}{1+4\alpha^2}. \tag{61}$$

These integrals will have limits which we will take to be $[-\alpha_0,\alpha_0]$ and $[-\Lambda_0^1,\Lambda_0^1]$ and we will see how these are determined by the concentrations of down spins and holes. Similarly from (54)

$$2\pi(J_{\gamma+1}^1-J_\gamma^1)=-\sum_{j=1}^{N-N^1}(\theta(2\Lambda_{\gamma+1}^1-2\alpha_j)-\theta(2\Lambda_\gamma^1-2\alpha_j)). \tag{62}$$

If Λ_γ^{1h} are the hole values of Λ^1, then

$$\rho_2(\Lambda^1)+\frac{2\pi}{N}\sum_{\gamma=1}^{N_h^1}\delta(\Lambda^1-\Lambda_\gamma^{1h})=\frac{2}{\pi}\int_{-\alpha_0}^{\alpha_0}\mathrm{d}\alpha\frac{\rho_1(\alpha)}{1+4(\Lambda^1-\alpha)^2} \tag{63}$$

where $\rho_2(\Lambda^1)$ is the continuum limit of $(2\pi/N)[1/(\Lambda_{\gamma+1}^1-\Lambda_\gamma^1)]$.

We recall that the index j in α_j lies in the interval $[1, N-N^1]$, and the index γ in Λ_γ^1 lies in the interval $[1, N-N^1-N^2]$. Hence

$$\int_{-\alpha_0}^{\alpha_0}\mathrm{d}\alpha\,\rho_1(\alpha)=\lim_{N\to\infty}\sum_j(\alpha_{j+1}-\alpha_j)\frac{2\pi}{N}\frac{1}{(\alpha_{j+1}-\alpha_j)}=\frac{2\pi(N-N^1)}{N} \tag{64}$$

and

$$\int_{-\Lambda_0^1}^{\Lambda_0^1}\mathrm{d}\Lambda^1\,\rho_2(\Lambda^1)=\lim_{N\to\infty}\sum_\gamma(\Lambda_{\gamma+1}^1-\Lambda_\gamma^1)\frac{2\pi}{N}\frac{1}{\Lambda_{\gamma+1}^1-\Lambda_\gamma^1}=\frac{2\pi(N-N^1-N^2)}{N}. \tag{65}$$

The energy E is

$$E=2(2N^1+N^2)-2N-2\sum_j\cos k_j. \tag{66}$$

Since

$$\cos k_j=\frac{1-\tan^2\frac{1}{2}k_j}{1+\tan^2\frac{1}{2}k_j}=\frac{1-4\alpha_j^2}{1+4\alpha_j^2} \tag{67}$$

then

$$\frac{E}{N}=2-\frac{1}{\pi}\int_{-\Lambda_0^1}^{\Lambda_0^1}\mathrm{d}\Lambda^1\,\rho_2(\Lambda^1)-\frac{1}{2\pi}\int_{-\alpha_0}^{\alpha_0}\mathrm{d}\alpha\frac{4\rho_1(\alpha)}{1+4\alpha^2}. \tag{68}$$

From (64) and (65) it is clear that

$$\Lambda_0^1=0\qquad\text{and}\qquad\alpha_0=\infty \tag{69}$$

corresponds to the half-filling case where the model reduces to the Heisenberg model. In general the integral equations (61) and (63) need to be solved numerically or through approximate application of Wiener–Hopf techniques (Andrei *et al* 1983). We will give an analytic treatment valid near half-filling, i.e. α_0 very large and Λ_0^1 very small. This will enable us to obtain limited information such as the gaplessness of the ground state. Let us first check what sort of half-filling is implied by (69). Equation (61) becomes

$$\frac{4}{1+4\alpha^2}=\rho_1(\alpha)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\rho_1(\alpha')}{1+(\alpha-\alpha')^2}\,\mathrm{d}\alpha'. \tag{70}$$

On writing

$$\rho_1(\alpha)=\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{\mathrm{i}p\alpha}\tilde{\rho}_1(p) \tag{71}$$

and on noting that

$$\frac{1}{\alpha^2+1}=\pi\int\frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{\mathrm{i}p\alpha}\,\mathrm{e}^{-|p|} \tag{72}$$

it is found that

$$\tilde{\rho}_1(p)=2\pi\frac{\mathrm{e}^{-(1/2)|p|}}{1+\mathrm{e}^{-|p|}}=\frac{\pi}{\cosh\frac{1}{2}|p|}. \tag{73}$$

Since

$$2\pi\frac{(N-N^1)}{N}=\int_{-\infty}^{\infty}\rho_1(\alpha)\,\mathrm{d}\alpha=\tilde{\rho}_1(0)=\pi \tag{74}$$

$$N^1/N=\tfrac{1}{2} \tag{75}$$

and so there are an equal number of up and down spins.

We shall consider the effect of introducing a small macroscopic number of real holes (as opposed to Bethe ansatz holes). Consequently Λ_0^1 will be small. Let us write

$$\rho_1(\alpha)=\rho_1^{(0)}(\alpha)+\rho_1^{(1)}(\alpha)+\dots \tag{76}$$

and

$$\rho_2(\Lambda^1)=\rho_2^{(0)}(\Lambda^1)+\rho_2^{(1)}(\Lambda^1)+\dots \tag{77}$$

$$\rho_2^{(0)}(\Lambda^1)=\frac{2}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha'\frac{\rho_1^{(0)}(\alpha')}{1+4(\Lambda^1-\alpha')^2} \tag{78}$$

where $\rho_1^{(1)}(\alpha)$ and $\rho_2^{(1)}(\Lambda^1)$ are small corrections due to doping.

From (61) in the absence of Bethe ansatz holes we obtain

$$\begin{aligned}\rho_1^{(1)}(\alpha)\simeq{}&\frac{1}{\pi}\int_{\alpha_0}^{\infty}\mathrm{d}\alpha'\frac{1}{1+(\alpha-\alpha')^2}\rho_1^{(0)}(\alpha')+\frac{1}{\pi}\int_{-\infty}^{-\alpha_0}\mathrm{d}\alpha'\frac{1}{1+(\alpha-\alpha')^2}\rho_1^{(0)}(\alpha')\\&-\frac{1}{\pi}\int_{-\alpha_0}^{\alpha_0}\mathrm{d}\alpha'\frac{1}{1+(\alpha-\alpha')^2}\rho_1^{(1)}(\alpha')+\frac{4}{\pi}\Lambda_0^1\rho_2^{(0)}(0)\frac{1}{1+4\alpha^2}.\end{aligned} \tag{79}$$

($\rho_1^{(1)}(\alpha)$ actually also depends implicitly on α_0 and Λ_0^1 and more properly should be written as $\rho_1^{(1)}(\alpha,\alpha_0,\Lambda_0^1)$.)

Since $\rho_1^{(1)}(\alpha')$ is small it is a good approximation to write

$$\int_{-\alpha_0}^{\alpha_0}\mathrm{d}\alpha'\frac{1}{1+(\alpha-\alpha')^2}\rho_1^{(1)}(\alpha')\sim\int_{-\infty}^{\infty}\mathrm{d}\alpha'\frac{1}{1+(\alpha-\alpha')^2}\rho_1^{(1)}(\alpha'). \tag{80}$$

(Similarly from (63) for completeness we note

$$\rho_2^{(1)}(\Lambda^1) \simeq -\frac{2}{\pi}\int_{\alpha_0}^{\infty} d\alpha \frac{\rho_1^{(0)}(\alpha)}{1+4(\Lambda^1-\alpha)^2} - \frac{2}{\pi}\int_{-\infty}^{-\alpha_0} d\alpha' \frac{1}{1+4(\Lambda^1-\alpha)^2} + \frac{2}{\pi}\int_{-\infty}^{\infty} d\alpha \frac{\rho_1^{(1)}(\alpha)}{1+4(\Lambda^1-\alpha)^2} \tag{81}$$

although we will not need the explicit form of $\rho_2^{(1)}(\Lambda^1)$ for our first-order calculation.)

On solving (79) by Fourier transformation we find

$$\rho_1^{(1)}(\alpha) = \frac{2}{\pi} e^{-2\pi\alpha_0}\left(\frac{1}{1+(\alpha_0-\alpha)^2} + \frac{1}{1+(\alpha_0+\alpha)^2}\right) + \frac{4\Lambda_0^1}{\pi}\rho_2^{(0)}(0)\frac{1}{1+4\alpha^2}. \tag{82}$$

The magnetization M is

$$M = \tfrac{1}{2}(N^1 - N^2) = \tfrac{1}{2}(N - 2(N-N^1) + (N - N^1 - N^2)) \tag{83}$$

and on using (82) we have

$$\frac{M}{N} = \frac{1}{2}\left(\frac{e^{-2\pi\alpha_0}}{\pi}\left(3 + \frac{1}{\pi\alpha_0}\right) + \frac{4}{\pi^2} e^{-\pi\alpha_0}\Lambda_0^1\rho_2^{(0)}(0)\right). \tag{84}$$

However α_0 is a function of Λ_0, i.e. given a certain doping level the spins align themselves in such a way so as to minimize the energy. We therefore need to calculate the energy. From (68) we have

$$\varepsilon \equiv \frac{E}{N} \simeq 2 - \frac{2}{\pi}\rho_2^{(0)}(0)\Lambda_0^1 - \frac{1}{2\pi}\int_{-\alpha_0}^{\alpha_0} d\alpha \frac{4\rho_1^{(0)}(\alpha)}{1+4\alpha^2} - \frac{1}{2\pi}\int_{-\alpha_0}^{\alpha_0} d\alpha \frac{4\rho_1^{(1)}(\alpha)}{1+4\alpha^2} \tag{85}$$

and so

$$\left.\frac{\partial\varepsilon}{\partial\alpha_0}\right|_{\Lambda_0^1} = -\frac{2}{\pi}\left(\frac{\rho_1^{(0)}(\alpha_0) + \rho_1^{(0)}(-\alpha_0) + \rho_1^{(1)}(\alpha_0, \alpha_0, \Lambda_0^1) + \rho_1^{(1)}(-\alpha_0, \alpha_0, \Lambda_0^1)}{1+4\alpha_0^2}\right) - \frac{2}{\pi}\int_{-\alpha_0}^{\alpha_0} d\alpha \frac{1}{1+4\alpha^2}\frac{\partial}{\partial\alpha_0}\rho_1^{(1)}(\alpha, \alpha_0, \Lambda_0^1). \tag{86}$$

After a certain amount of analysis it is possible to show that

$$\frac{\partial\varepsilon}{\partial\alpha_0} = -\frac{1}{\pi\alpha^2}\left(\frac{2\pi}{\cosh(2\pi\alpha_0)} + \frac{4}{\pi} e^{-2\pi\alpha_0}(f(0) + f(2\alpha_0)) + \Lambda_0^1\rho_2^{(0)}(0)\right) - \frac{8}{\pi} e^{-2\pi\alpha_0}(-2\pi g(\alpha_0) + g'(\alpha_0)) \tag{87}$$

where

$$f(\alpha) = \tfrac{1}{2}\sum_{r=1}^{\infty}(-1)^{r+1}\frac{r}{r^2+\alpha^2} \tag{88}$$

and

$$g(\alpha) = \tfrac{1}{2}\sum_{r=1}^{\infty}(-1)^{r+1}\frac{r+\frac{1}{2}}{(r+\frac{1}{2})^2+\alpha^2}. \tag{89}$$

Using asymptotic estimates for f and g we can deduce

$$\frac{\partial\varepsilon}{\partial\alpha_0} < 0 \tag{90}$$

and so the minimum of energy is found for $\alpha_0=\infty$. Equation (84) then implies that

$$M=0 \tag{91}$$

in the ground state.

In order to consider excited states we have to examine the effect of Bethe ansatz holes (Andrei *et al* 1983). Now we let

$$\rho_1(\alpha)\to\rho_1(\alpha)+\Delta\rho_1(\alpha) \tag{92}$$

and

$$\rho_2(\Lambda^1)\to\rho_2(\Lambda^1)+\Delta\rho_2(\Lambda^1) \tag{93}$$

where $\Delta\rho_1(\alpha)$ and $\Delta\rho_2(\Lambda^1)$ are changes in ρ_1 and ρ_2 due to the presence of Bethe ansatz holes. Clearly from (61) and (63)

$$\Delta\rho_1(\alpha)+\frac{2\pi}{N}\sum_{j=1}^{N_h}\delta(\alpha-\alpha_j^h)$$
$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Delta\rho_1(\alpha')}{1+(\alpha-\alpha')^2}\,d\alpha'+\frac{2}{\pi}\int_{-\Lambda_0^1}^{\Lambda_0^1}\frac{\Delta\rho_2(\Lambda^1)}{1+4(\alpha-\Lambda^1)^2}\,d\Lambda^1 \tag{94}$$

and

$$\Delta\rho_2(\Lambda^1)+\frac{2\pi}{N}\sum_{\gamma=1}^{N_h^1}\delta(\Lambda^1-\Lambda_\gamma^{1h})=\frac{2}{\pi}\int_{-\infty}^{\infty}d\alpha\,\frac{\Delta\rho_1(\alpha)}{1+4(\Lambda^1-\alpha)^2}. \tag{95}$$

These equations can be solved by Fourier transforming. We obtain

$$\Delta\tilde\rho_1(p)=\frac{\Lambda_0^1\Delta\rho_2(0)}{\cosh\frac{1}{2}p}-\frac{2\pi}{N}\frac{\sum_{j=1}^{N_h}e^{ip\alpha_j^h}}{1+e^{-|p|}} \tag{96}$$

where $\Delta\tilde\rho_1$ is the Fourier transform of $\Delta\rho_1$. Similarly

$$\Delta\tilde\rho_2(p)=e^{-(1/2)|p|}\Delta\tilde\rho_1(p)-\frac{2\pi}{N}\sum_{\gamma=1}^{N_h^1}e^{-ip\Lambda_\gamma^{1h}}. \tag{97}$$

Consequently, since

$$\Delta\rho_2(0)=\int\frac{dp}{2\pi}\,\Delta\tilde\rho_2(p) \tag{98}$$

we have

$$\Delta\rho_2(0)=-\frac{2\pi}{N}\sum_{j=1}^{N_h}\frac{1}{\cosh(2\pi\alpha_j^h)}-\frac{2\pi}{N}\sum_{\gamma=1}^{N_h^1}\delta(\Lambda_\gamma^{1h})\left(1-\frac{2}{\pi}(\log 2)\Lambda_0^1\right)^{-1}. \tag{99}$$

We can now calculate the change in energy ΔE due to the Bethe ansatz holes. From (68), (85) and (96) we have

$$\frac{\Delta E}{N}=\frac{2\pi}{N}\left[\left(\frac{2\log 2}{\pi}\Lambda_0^1+1\right)\sum_{j=1}^{N_h}\frac{1}{\cosh(2\pi\alpha_j^h)}+\frac{2\log 2}{\pi}\Lambda_0^1\sum_{\gamma=1}^{N_h^1}\frac{\delta(\Lambda_\gamma^{1h})}{N}\right]-\frac{2}{\pi}\Lambda_0^1\Delta\rho_2(0). \tag{100}$$

The change in the magnetization ΔM by definition is

$$\Delta M=\tfrac{1}{2}(-2\Delta(N-N^1)+\Delta(N-N^1-N^2)) \tag{101}$$

which on using

$$\frac{2\pi}{N}\Delta(N-N^1)=\int_{-\infty}^{\infty}\Delta\rho_1(\alpha)\,\mathrm{d}\alpha \tag{102}$$

and

$$\frac{2\pi}{N}\Delta(N-N^1-N^2)\simeq 2\Lambda_0^1\Delta\rho_2(0) \tag{103}$$

gives

$$\Delta M=\tfrac{1}{2}N_\mathrm{h} \tag{104}$$

and

$$\Delta(N-N^1)=-\tfrac{1}{2}N_\mathrm{h}-\Lambda_0^1\left(\sum_{j=1}^{N_\mathrm{h}}\frac{1}{\cosh(2\pi\alpha_j^\mathrm{h})}+\sum_{\gamma=1}^{N_\mathrm{h}^1}\delta(\Lambda_\gamma^{1\mathrm{h}})\right). \tag{105}$$

The gap above the ground state in (100) is zero since $[\cosh(2\pi\alpha_j^\mathrm{h})]^{-1}$ can be chosen to be arbitrarily small (or α_j^h arbitrarily large) and Λ_h^1 taken to be non-zero. For these same conditions $\Delta(N-N^1)$ is $-\frac{1}{2}N_\mathrm{h}$ which has to be an integer. Consequently the least complicated zero-energy excitation that has been constructed has angular momentum 1. We have thus obtained valuable information from (61) and (63) with our simple approximation. Our analysis of the $t-J$ model bears throughout a strong resemblance to that for the Heisenberg model. Many generalizations of the latter are possible but for both physical and mathematical reasons the supersymmetric generalization that we have considered is a particularly non-trivial one.

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